

A PSEUDO-DIFFERENTIAL CALCULUS ON THE HEISENBERG GROUP

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ABSTRACT. In this note we present a symbolic pseudo-differential calculus on the Heisenberg group. We particularise to this group our general construction [4, 3, 2] of pseudo-differential calculi on graded groups. The relation between the Weyl quantization and the representations of the Heisenberg group enables us to consider here scalar-valued symbols. We find that the conditions defining the symbol classes are similar but different to the ones in [1]. Applications are given to Schwartz hypoellipticity and to subelliptic estimates on the Heisenberg group.

Key words: Harmonic Analysis, Heisenberg group, pseudo-differential operators.

MSC classes: 35S05, 43A80

1. INTRODUCTION

In [4], see also [3, 2], a pseudo-differential calculus is developed in the setting of graded Lie groups using their representations. Here we present the results of this construction in the particular case of the Heisenberg group \mathbb{H}_n .

It is well known that the representations of \mathbb{H}_n are intimately linked with the Weyl quantization on \mathbb{R}^n (see e.g., [10], and Section 2 below). Together with the analogue of the Kohn-Nirenberg quantization on Lie groups (see e.g., [10, 7, 4], and Section 4 below), this link enables the development of pseudo-differential calculi on \mathbb{H}_n with scalar-valued symbols that depend on parameters. However, the remaining difficulty lies in finding conditions to be imposed on those symbols so that the resulting class of operators has the expected properties of a calculus.

Although M. Taylor explained these general ideas in the setting of the Heisenberg groups in [10], he chose to restrict his analysis in [10] mainly to invariant (i.e. convolution) operators on \mathbb{H}_n with symbols defined by some asymptotic expansions. To the authors' knowledge, the only study of non-invariant calculi with scalar-valued symbols on \mathbb{H}_n was done, until now, by H. Bahouri, C. Fermanian-Kammerer and I. Gallagher in [1]. Their work is devoted to the case of \mathbb{H}_n only. Moreover, the conditions imposed on the scalar-valued symbols might appear difficult to apprehend for some readers, as they come from technical parts of the proofs of the calculi's properties (see the more recent version of [1] on the server Hal). Our conditions on symbol classes differ from those in [1] for small λ . At the end of this note we

list several applications of the analysis in our classes, to the hypoellipticity properties and subelliptic estimates for several operators on the Heisenberg group.

Our approach to find the conditions on the symbols is different from [10] and [1]: we particularise to the setting of \mathbb{H}_n our definition of pseudo-differential calculi valid on a large class of nilpotent Lie groups, namely the graded groups, see [4, 3, 2]. In our general construction, the symbols are operator-valued. Nonetheless on \mathbb{H}_n , using the link between the Weyl quantization and the representations of \mathbb{H}_n , this is equivalent to using the scalar-valued symbols. The purpose of this note is to present what the general conditions on the symbols given in [4, 3, 2] become when expressed on the level of scalar-valued symbols of \mathbb{H}_n . In particular, we find conditions which are similar but different to the ones in [1]. As applications for our analysis, we give sufficient condition for Schwartz hypoellipticity and for subelliptic estimates on the Heisenberg group.

2. SCHRÖDINGER REPRESENTATIONS AND WEYL QUANTIZATION

We start by fixing the notation required for presenting our results. We realise the Heisenberg group \mathbb{H}_n as the manifold \mathbb{R}^{2n+1} endowed with the law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)),$$

where (x, y, t) and (x', y', t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$. Here we adopt the convention that if x and y are two vectors in \mathbb{R}^n for some $n \in \mathbb{N}$, then xy denotes their standard scalar product

$$xy = \sum_{j=1}^n x_j y_j \quad \text{if} \quad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n).$$

The canonical basis for the Lie algebra \mathfrak{h}_n of \mathbb{H}_n is given by the left-invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1, \dots, n, \quad \text{and} \quad T = \partial_t.$$

The canonical commutation relations are

$$[X_j, Y_j] = T, \quad j = 1, \dots, n,$$

and T is the centre of \mathfrak{h}_n . The Heisenberg Lie algebra is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$ where V_1 is linearly spanned by the X_j 's and Y_j 's, while $V_2 = \mathbb{R}T$. Therefore, the group \mathbb{H}_n is naturally equipped with the family of dilations D_r given by

$$D_r(x, y, t) = r(x, y, t) = (rx, ry, r^2t), \quad (x, y, t) \in \mathbb{H}_n, \quad r > 0.$$

The ‘canonical’ positive Rockland operator in this setting is $\mathcal{R} = -\mathcal{L}$, where \mathcal{L} is the sub-Laplacian

$$\mathcal{L} := \sum_{j=1}^n (X_j^2 + Y_j^2) = \sum_{j=1}^n \left(\left(\partial_{x_j} - \frac{y_j}{2} \partial_t \right)^2 + \left(\partial_{y_j} + \frac{x_j}{2} \partial_t \right)^2 \right).$$

The Schrödinger representations of the Heisenberg group \mathbb{H}_n are the infinite dimensional unitary representations of \mathbb{H}_n (we allow ourselves to identify unitary representations with their unitary equivalence classes). Parametrised by $\lambda \in \mathbb{R} \setminus \{0\}$, they act on $L^2(\mathbb{R}^n)$. We denote them by π_λ and realise them as

$$(1) \quad \pi_\lambda(x, y, t)h(u) = e^{i\lambda(t + \frac{1}{2}xy)} e^{i\sqrt{\lambda}yu} h(u + \sqrt{|\lambda|}x),$$

for $h \in L^2(\mathbb{R}^n)$, $u \in \mathbb{R}^n$, and $(x, y, t) \in \mathbb{H}_n$, where we use the convention

$$(2) \quad \sqrt{\lambda} := \operatorname{sgn}(\lambda) \sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0. \end{cases}$$

The group Fourier transform of a function $\kappa \in L^1(\mathbb{H}_n)$ is by definition

$$\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) := \int_{\mathbb{H}_n} \kappa(x, y, t) \pi_\lambda(x, y, t)^* dx dy dt.$$

As already noted in [10], it can be effectively computed by

$$\pi_\lambda(\kappa)h(u) = \int_{\mathbb{R}^{2n+1}} \kappa(x, y, t) e^{i\lambda(-t + \frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu} h(u - \sqrt{|\lambda|}x) dx dy dt,$$

for $h \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$, that is,

$$(3) \quad \widehat{\kappa}(\pi_\lambda)(u) = \pi_\lambda(\kappa)h(u) = (2\pi)^{\frac{n}{2}} \operatorname{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|} \cdot, \sqrt{\lambda} \cdot, \lambda) \right].$$

Here the Fourier transform $\mathcal{F}_{\mathbb{R}^{2n+1}} = \mathcal{F}_{\mathbb{R}^N}$ is defined via

$$\mathcal{F}_{\mathbb{R}^N} f(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x) e^{-ix\xi} dx, \quad (\xi \in \mathbb{R}^N, f \in L^1(\mathbb{R}^N)),$$

and Op^W denotes the Weyl quantization, which is given for a reasonable symbol a on $\mathbb{R}^n \times \mathbb{R}^n$, by

$$\operatorname{Op}^W(a)f(u) = a(D, X)f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) f(v) dv d\xi,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. We keep the same notation π_λ for the corresponding infinitesimal representations. We readily compute that

$$\begin{aligned} \pi_\lambda(X_j) &= \sqrt{|\lambda|} \partial_{u_j} = \operatorname{Op}^W \left(i\sqrt{|\lambda|} \xi_j \right), \\ \pi_\lambda(Y_j) &= i\sqrt{\lambda} u_j = \operatorname{Op}^W \left(i\sqrt{\lambda} u_j \right), \\ \pi_\lambda(T) &= i\lambda I = \operatorname{Op}^W(i\lambda), \end{aligned}$$

thus

$$\pi_\lambda(\mathcal{L}) = \sum_{j=1}^n (\pi_\lambda(X_j)^2 + \pi_\lambda(Y_j)^2) = |\lambda| \sum_{j=1}^n (\partial_{u_j}^2 - u_j^2) = -\text{Op}^W \left(|\lambda| \sum_{j=1}^n (\xi_j^2 + u_j^2) \right).$$

With our choice of notation and definitions, the Plancherel measure is $c_n |\lambda|^n d\lambda$ in the sense that the Plancherel formula

$$(4) \quad \int_{\mathbb{H}_{n_o}} |\kappa(x, y, t)|^2 dx dy dt = c_n \int_{\mathbb{R} \setminus \{0\}} \|\pi_\lambda(\kappa)\|_{\text{HS}}^2 |\lambda|^n d\lambda.$$

holds for any $\kappa \in \mathcal{S}(\mathbb{H}_n)$. For the value of the constant c_n , see [4]. Here $\|\cdot\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm of an operator on $L^2(\mathbb{R}^n)$, that is, $\|B\|_{\text{HS}}^2 = \text{Tr}(B^* B)$. This allows one to extend unitarily the definition of the group Fourier transform to $L^2(\mathbb{H}_n)$. Formula (4) then holds for any $\kappa \in L^2(\mathbb{H}_n)$.

3. DIFFERENCE OPERATORS

Difference operators were defined in [7, 9] as acting on Fourier coefficients on compact Lie groups, and on graded nilpotent Lie groups in [4]. In the setting of the Heisenberg group, this yields the definition of the difference operators Δ_{x_j} , Δ_{y_j} , and Δ_t via

$$\Delta_{x_j} \widehat{\kappa}(\pi_\lambda) := \pi_\lambda(x_j \kappa), \quad \Delta_{y_j} \widehat{\kappa}(\pi_\lambda) := \pi_\lambda(y_j \kappa), \quad \Delta_t \widehat{\kappa}(\pi_\lambda) := \pi_\lambda(t \kappa),$$

for suitable distributions κ defined on \mathbb{H}_n . We can compute that

$$\begin{aligned} \Delta_{x_j} |_{\pi_\lambda} &= \frac{1}{i\lambda} \text{ad}(\pi_\lambda(Y_j)) = \frac{1}{\sqrt{|\lambda|}} \text{ad} u_j, \\ \Delta_{y_j} |_{\pi_\lambda} &= -\frac{1}{i\lambda} \text{ad}(\pi_\lambda(X_j)) = -\frac{1}{i\sqrt{\lambda}} \text{ad} \partial_{u_j}, \end{aligned}$$

and

$$\Delta_t |_{\pi_\lambda} = i\partial_\lambda + \frac{1}{2} \sum_{j=1}^n \Delta_{x_j} \Delta_{y_j} |_{\pi_\lambda} - \frac{i}{2\lambda} \sum_{j=1}^n (\pi_\lambda(Y_j) \Delta_{y_j} |_{\pi_\lambda} + \Delta_{x_j} |_{\pi_\lambda} \pi_\lambda(X_j)).$$

When $\pi_\lambda(\kappa) = \text{Op}^W(a_\lambda)$ and $a_\lambda = \{a_\lambda(\xi, u)\}$, we have

$$(5) \quad \left. \begin{aligned} \Delta_{x_j} \pi_\lambda(\kappa) &= \text{Op}^W \left(\frac{i}{\sqrt{|\lambda|}} \partial_{\xi_j} a_\lambda \right) \\ \Delta_{y_j} \pi_\lambda(\kappa) &= \text{Op}^W \left(\frac{i}{\sqrt{\lambda}} \partial_{u_j} a_\lambda \right) \\ \Delta_t \pi_\lambda(\kappa) &= i \text{Op}^W \left(\tilde{\partial}_{\lambda, \xi, u} a_\lambda(\xi, u) \right) \end{aligned} \right\}$$

where

$$(6) \quad \tilde{\partial}_{\lambda, \xi, u} := \partial_\lambda - \frac{1}{2\lambda} \sum_{j=1}^n (u_j \partial_{u_j} + \xi_j \partial_{\xi_j}).$$

For example, we have

$$\begin{aligned}\Delta_{x_j}\pi_\lambda(Y_k) &= \Delta_{x_j}\pi_\lambda(T) = \Delta_{y_j}\pi_\lambda(X_k) = \Delta_{y_j}\pi_\lambda(T) = 0, \\ \Delta_{x_j}\pi_\lambda(X_k) &= \Delta_{y_j}\pi_\lambda(Y_k) = -\delta_{jk}\mathbf{I}, \quad \Delta_t\pi_\lambda(T) = -\mathbf{I}, \\ \Delta_{x_j}\pi_\lambda(\mathcal{L}) &= -2\pi_\lambda(X_j), \quad \Delta_{y_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(Y_j), \quad \Delta_t\pi_\lambda(\mathcal{L}) = 0.\end{aligned}$$

The following equalities shed some light on why, for example in [1], another normalisation of the Weyl symbol is preferred. Indeed, the expressions on the right-hand sides in (5), in particular for the operator $\tilde{\partial}_{\lambda,\xi,u}$ defined in (6), become very simple.

Lemma 3.1. *Let $a_\lambda = \{a_\lambda(\xi, u)\}$ be a family of Weyl symbols depending smoothly on $\lambda \neq 0$. If \tilde{a}_λ is the renormalisation obtained via*

$$a_\lambda(\xi, u) := \tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u),$$

then

$$\begin{aligned}\tilde{\partial}_{\lambda,\xi,u}a_\lambda(\xi, u) &= \{\partial_\lambda \tilde{a}_\lambda\}(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u), \\ \frac{1}{i\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda &= (\partial_{\xi_j}\tilde{a}_\lambda)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u), \quad \text{and} \quad \frac{1}{i\sqrt{\lambda}}\partial_{u_j}a_\lambda = (\partial_{u_j}\tilde{a}_\lambda)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u).\end{aligned}$$

Consequently,

$$\Delta_{x_j}\pi_\lambda(\kappa) = i\text{Op}^W(\partial_{\xi_j}\tilde{a}_\lambda), \quad \Delta_{y_j}\pi_\lambda(\kappa) = i\text{Op}^W(\partial_{u_j}\tilde{a}_\lambda) \quad \text{and} \quad \Delta_t\pi_\lambda(\kappa) = i\text{Op}^W(\partial_\lambda\tilde{a}_\lambda).$$

4. QUANTIZATION AND SYMBOL CLASSES

In this note, for simplicity, we change slightly the notation with respect to the general case developed in [4]. Firstly we want to keep the letter x to denote part of the coordinates of the Heisenberg group and we choose to denote a general element of the Heisenberg group by, e.g.,

$$g = (x, y, t) \in \mathbb{H}_n.$$

Secondly we may define a symbol as parametrised by

$$\sigma(g, \lambda) := \sigma(g, \pi_\lambda), \quad (g, \lambda) \in \mathbb{H}_n \times \mathbb{R} \setminus \{0\}.$$

Thirdly we modify the indices $\alpha \in \mathbb{N}_0^{2n+1}$ in order to write them as

$$\alpha = (\alpha_1, \alpha_2, \alpha_3),$$

with

$$\alpha_1 = (\alpha_{1,1}, \dots, \alpha_{1,n}) \in \mathbb{N}_0^n, \quad \alpha_2 = (\alpha_{2,1}, \dots, \alpha_{2,n}) \in \mathbb{N}_0^n, \quad \alpha_3 \in \mathbb{N}_0.$$

The homogeneous degree of α is then

$$[\alpha] = |\alpha_1| + |\alpha_2| + 2\alpha_3.$$

For each α we write

$$g^\alpha = x^{\alpha_1} y^{\alpha_2} t^{\alpha_3}, \quad \text{where} \quad x^{\alpha_1} = x_1^{\alpha_{11}} \dots x_n^{\alpha_{1n}}, \quad y^{\alpha_2} = y_1^{\alpha_{21}} \dots y_n^{\alpha_{2n}},$$

and we define the corresponding difference operator

$$\Delta'^\alpha := \Delta_x^{\alpha_1} \Delta_y^{\alpha_2} \Delta_t^{\alpha_3}, \quad \text{where} \quad \Delta_x^{\alpha_1} := \Delta_{x_1}^{\alpha_{11}} \dots \Delta_{x_n}^{\alpha_{1n}}, \quad \Delta_y^{\alpha_2} := \Delta_{y_1}^{\alpha_{21}} \dots \Delta_{y_n}^{\alpha_{2n}}.$$

We also write $X^\alpha = X^{\alpha_1} Y^{\alpha_2} T^{\alpha_3}$, where $X^{\alpha_1} = X_1^{\alpha_{11}} \dots X_n^{\alpha_{1n}}$, and $Y^{\alpha_2} = Y_1^{\alpha_{21}} \dots Y_n^{\alpha_{2n}}$.

Following [4], we define the symbol class $S_{\rho,\delta}^m(\mathbb{H}_n)$ as the set of symbols σ for which all the following quantities are finite:

$$\|\sigma\|_{S_{\rho,\delta}^m,a,b,c} := \sup_{\lambda \in \mathbb{R} \setminus \{0\}, g \in \mathbb{H}_n} \|\sigma(g, \lambda)\|_{S_{\rho,\delta}^m,a,b,c}, \quad a, b, c \in \mathbb{N}_0,$$

where

$$\|\sigma(g, \lambda)\|_{S_{\rho,\delta}^m,a,b,c} := \sup_{\substack{[\alpha] \leq a \\ [\beta] \leq b, |\gamma| \leq c}} \|\pi_\lambda(I - \mathcal{L})^{\frac{\rho[\alpha] - m - \delta[\beta] + \gamma}{2}} X_g^\beta \Delta'^\alpha \sigma(g, \lambda) \pi_\lambda(I - \mathcal{L})^{-\frac{\gamma}{2}}\|_{op}.$$

A natural quantization on any type I Lie group is the analogue of the Kohn-Nirenberg quantization on \mathbb{R}^n , see, e.g., [10] for general remarks, [7] for the consistent development in the case of compact Lie group, and [4] for the case of nilpotent Lie groups. In the particular case of the Heisenberg group, this quantization associates to a symbol σ (for example in $S_{\rho,\delta}^m(\mathbb{H}_n)$) the operator $A = \text{Op}(\sigma)$ acting on $\mathcal{S}(\mathbb{H}_n)$ given by

$$(7) \quad A\varphi(g) = c_n \int_{\lambda \in \mathbb{R} \setminus \{0\}} \text{Tr}(\pi_\lambda(g) \sigma(g, \lambda) \pi_\lambda(\varphi)) |\lambda|^n d\lambda.$$

Here we have used our notation, especially for the Plancherel measure $c_n |\lambda|^n d\lambda$, see (4). We denote by

$$(8) \quad \Psi_{\rho,\delta}^m(\mathbb{H}_n) := \{\text{Op}(\sigma), \sigma \in S_{\rho,\delta}^m(\mathbb{H}_n)\},$$

the class of operators corresponding to the symbols in $S_{\rho,\delta}^m(\mathbb{H}_n)$ via this quantization.

The main result of this note shows that the symbols $\sigma = \{\sigma(g, \lambda)\}$ in $S_{\rho,\delta}^m$ are all of the form $\sigma(g, \lambda) = \text{Op}^W(a_{\lambda,g}(\xi, u))$ with $a_{\lambda,g}$ (called λ -symbols) satisfying properties of Shubin type:

Theorem 4.1. *Let ρ, δ be real numbers such that $1 \geq \rho \geq \delta \geq 0$ and $(\rho, \delta) \neq 0$.*

- (1) *If $\sigma = \{\sigma(g, \lambda)\}$ is in $S_{\rho,\delta}^m(\mathbb{H}_n)$ then there exists a unique smooth function $a = \{a(g, \lambda, \xi, u) = a_{g,\lambda}(\xi, u)\}$ on $\mathbb{H}_n \times \mathbb{R} \setminus \{0\} \times \mathbb{R}^n \times \mathbb{R}^n$ such that*

$$(9) \quad \sigma(g, \lambda) = \text{Op}^W(a_{g,\lambda}).$$

(10) *It satisfies for any $\alpha, \beta \in \mathbb{N}_0^n$, $\tilde{\beta} \in \mathbb{N}_0^{2n+1}$ and $\tilde{\alpha} \in \mathbb{N}_0$,*

$$|\partial_\xi^\alpha \partial_u^\beta \tilde{\partial}_{\lambda, \xi, u}^{\tilde{\alpha}} X_g^{\tilde{\beta}} a_{g, \lambda}(\xi, u)| \leq C_{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}} |\lambda|^{\rho \frac{|\alpha| + |\beta|}{2}} (1 + |\lambda|(1 + |\xi|^2 + |u|^2))^{\frac{m - 2\rho\tilde{\alpha} + \delta[\tilde{\beta}] - \rho(|\alpha| + |\beta|)}{2}},$$

where the operator $\tilde{\partial}_{\lambda, \xi, u}$ was defined in (6).

- (2) *Conversely, if $a = \{a(g, \lambda, \xi, u) = a_{g, \lambda}(\xi, u)\}$ is a smooth function on $\mathbb{H}_n \times \mathbb{R} \setminus \{0\} \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying (10) for every $\alpha, \beta \in \mathbb{N}_0^n$, $\tilde{\alpha} \in \mathbb{N}_0$, then there exists a unique symbol $\sigma \in S_{\rho, \delta}^m(\mathbb{H}_n)$ such that (9) holds.*
- (3) *The resulting class of operators $\cup_{m \in \mathbb{R}} \Psi_{\rho, \delta}^m(\mathbb{H}_n)$ is an algebra of operators, the product being the usual composition. It is stable under taking the adjoint and contains the left-invariant differential calculus. Each operator $A \in \Psi_{\rho, \delta}^m(\mathbb{H}_n)$ maps continuously the Sobolev space $L_s^2(\mathbb{H}_n)$ of the Heisenberg group to $L_{s-m}^2(\mathbb{H}_n)$ with the loss of m derivatives (for any $s \in \mathbb{R}$).*

For the definition of the Sobolev spaces $L_s^2(\mathbb{H}_n)$ on \mathbb{H}_n and more generally on any stratified group, see [5]. Part (iii) summarises the main results of the general construction made in [4] on any graded groups.

Let us notice that writing $\sigma(g, \lambda) = \text{Op}^W(a_{g, \lambda})$ as in (9) and using (3) for

$$\pi_\lambda(\varphi)\pi_\lambda(g) = \pi_\lambda(\varphi(g \cdot))$$

yield the following alternative formula for the quantization given in (7):

$$(11) \quad A\varphi(g) = c'_n \int_{\lambda \in \mathbb{R} \setminus \{0\}} \text{Tr} \left(\text{Op}^W(a_{g, \lambda}) \text{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\varphi(g \cdot))(\sqrt{|\lambda|} \cdot, \sqrt{\lambda} \cdot, \lambda) \right] \right) |\lambda|^n d\lambda.$$

For the value of the constant c'_n , see [4]. The formula (11) now involves mainly ‘Euclidean objects’.

5. SOME APPLICATIONS

We finally note several applications of the above theorems to questions of hypoellipticity of (pseudo)differential operators on the Heisenberg group. We say that a pseudo-differential operator A is *Schwartz hypoelliptic* whenever $f \in \mathcal{S}'(\mathbb{H}_n)$, $Af \in \mathcal{S}(\mathbb{H}_n)$ imply that $f \in \mathcal{S}(\mathbb{H}_n)$. Then, for example, as a simple consequence of our calculus we obtain that the operator $I - \mathcal{L}$ is Schwartz hypoelliptic. In fact, criteria can be given in terms of the λ -symbols:

Corollary 5.1. *Let $m \in \mathbb{R}$ and $1 \geq \rho \geq \delta \geq 0$, $\rho \neq 0$. Let $\sigma = \{\sigma(g, \lambda)\}$ be in $S_{\rho, \delta}^m(\mathbb{H}_n)$ with $\sigma(g, \lambda) = \text{Op}^W(a_{g, \lambda})$ as in Theorem 4.1. Assume that there are $R \in \mathbb{R}$ and $C > 0$ such that for any $(\xi, u) \in \mathbb{R}^{2n}$ and $\lambda \neq 0$ satisfying $|\lambda|(|\xi|^2 + |u|^2) \geq R$ we have*

$$(12) \quad |a_{g, \lambda}(\xi, u)| \geq C (1 + |\lambda|(1 + |\xi|^2 + |u|^2))^{\frac{m}{2}}.$$

Then the operator A in (7) (or, alternatively, in (11)) has a left parametrix, i.e. there exists $B \in \Psi_{\rho,\delta}^{-m}(\mathbb{H}_n)$ such that $BA - I \in \Psi^{-\infty}$.

Corollary 5.1 has also a corresponding ‘hypoellipticity version’ which we omit here, but we give a few examples of both of them. First, let $m, m_o \in 2\mathbb{N}$ be two even integers such that $m \geq m_o$. Let A be a differential operator given by either $X^m + iY^{m_o} + T^{m_o/2}$ or $X^{m_o} + iY^m + T^{m_o/2}$ on \mathbb{H}_1 . Then A is Schwartz hypoelliptic and satisfies the subelliptic estimates

$$\forall s \in \mathbb{R} \quad \exists C > 0 \quad \forall f \in \mathcal{S}(\mathbb{H}_1) \quad \|f\|_{L_{s+m_o}^p(\mathbb{H}_1)} \leq C \|Af\|_{L_s^p(\mathbb{H}_1)}.$$

The above mentioned conclusion that $I - \mathcal{L}$ is Schwartz hypoelliptic can be also extended to variable coefficients using our calculus. For example, if f_1 and f_2 are complex-valued smooth functions on \mathbb{H}_n such that

$$\inf_{x \in \mathbb{H}_n, \lambda \geq \Lambda} \frac{|f_1(x) + f_2(x)\lambda|}{1 + \lambda} > 0 \quad \text{for some } \Lambda \geq 0,$$

and such that functions $X^{\alpha_1}f_1$, $X^{\alpha_2}f_2$ are bounded for every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$, then the differential operator $f_1(x) - f_2(x)\mathcal{L}$ is Schwartz-hypoelliptic and satisfies the following subelliptic estimates

$$\forall s \in \mathbb{R} \quad \exists C > 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{H}_n) \quad \|\varphi\|_{L_{s+2}^p(\mathbb{H}_n)} \leq C \|f_1\varphi - f_2\mathcal{L}\varphi\|_{L_s^p(\mathbb{H}_n)}.$$

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